**III YEAR - V SEMESTER**
**COURSE CODE: 7BMA5C1**

# CORE COURSE - IX – REAL ANALYSIS

**Unit – I**

Introduction – Sets and functions – Countable and Uncountable sets – Inequalities of Holder and Minkowski – Metric spaces – Definition and examples – Bounded sets in a metric space – Open Ball in a metric space – Opensets.

**Unit – II**

Subspace – Interior of a set – Closed sets – Closure – limit point – Dense sets – Completeness – Baire's Category Theorem

**Unit – III**

Continuity – Homeomorphism – Uniform continuity.

**Unit – IV**

Connectedness – Definition and examples – Connected subsets of R – Connectedness & Continuity.

**Unit – V**

Compact Metric spaces – Compact subsets of R – Equivalent Characterization for Compactness – Compactness and Continuity.

**Text Book:**

1. Modern Analysis, Dr. S.Arumugam& Mr. A.ThangapandiIssac, New Gamma Publishing House, Palayamkottai, Edition 2015.

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|---|---|
| **Unit I** | Chapter 1sections 1.1 to 1.4<br>Chapter 2 sections 2.1 to 2.4 |
| **Unit II** | Chapter 2 sections 2.5 to 2.10 & Chapter 3 |
| **Unit III** | Chapter 4 sections 4.1 to 4.3 |
| **Unit IV** | Chapter 5 |
| **Unit V** | Chapter 6 |

**Book for Reference:**

1. Richard R.Goldberg, Methods of Real analysis, IBM Publishing, New Delhi.

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Problem 3 :

If $d$ is a metric on $M$ is $d^2$ a metric on $M$?

Soln: Consider $d(x,y)$ defined on $R$ by

$d(x,y) = |x-y|$

We know that $d$ is metric on $R$

$d^2(x,y) = |x-y|^2 = (x-y)^2$

But $d^2$ is not a metric (refer problem 2)

Problem 4.

If $d$ is a metric on $M$. Prove that $\sqrt{d}$ is a metric on $M$.

Soln: Let $x,y,z \in M$

$\therefore$ We have $\sqrt{d(x,y)} \geq 0$ (since $d(x,y) \geq 0$)

Also $\sqrt{d(x,y)} = \sqrt{d(y,x)}$

Now, $d(x,z) \leq d(x,y) + d(y,z)$

$\therefore \sqrt{d(x,z)} \leq \sqrt{d(x,y) + d(y,z)}$

$\leq \sqrt{d(x,y)} + \sqrt{d(y,z)}$ Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$

Hence $\sqrt{d}$ is a metric on $M$.

Problem 5 :

Let $(M,d)$ be a metric space. define $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. Prove that $d_1$ is a metric on $M$.

Soln: $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \geq 0$ (since $d(x,y) \geq 0$

$d(x,y) = 0$ iff $x = y$

and $d(x,y) = d(y,x)$

Now, let $x, y, z \in l_p$

Taking, $a_i = x_i - y_i$ and $b_i = y_i - z_i$ in (a) we get

$$\left(\sum_{i=1}^{\infty} |x_i - z_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i - z_i|^p\right)^{1/p}$$

$d(x,z) \leq d(x,y) + d(y,z)$.

Hence $d$ is a metric on $l_p$.

Example 10.

Let $M$ be the set of all bounded real valued functions define on a non-empty set $E$. Define $d(f,g) = \sup\{|f(x) - g(x)| / x \in E\}$

$d$ is a metric on $M$.

Proof: $d(f,g) = \sup\{|f(x) - g(x)|\} \geq 0$

Also, $d(f,g) = 0 \Leftrightarrow \sup\{|f(x) - g(x)|\} \geq 0$

$\Leftrightarrow |f(x) - g(x)| = 0$ for all $x \in E$

$\Leftrightarrow f(x) = g(x)$ for all $x \in E$

$\Leftrightarrow f = g$

Also, $d(f,g) = \sup\{|f(x) - g(x)|\}$

$= \sup\{|g(x) - f(x)|\}$

$= d(g,f)$

Now, let $f, g, h \in M$

**Note** Let $A$ be a countably infinite set then there is a bijection $f$ from $N$ to $A$. Let $f(1) = a_1, f(2) = a_2, \ldots, f(n) = a_n$.

Then $A = \{a_1, a_2, \ldots, a_n, \ldots\}$

Thus all the elements of $A$ can be labelled by using the elements of $N$.

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Example 1

$\{2, 4, 6, \ldots, 2n\}$ is a countable set

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Example 2

$\mathbb{Z}$ is countable (refer example 2)

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Example 3

Let $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$

The function $f : N \to A$ defined by $f(n) = \frac{n}{n+1}$ is a bijection

Hence $A$ is countable

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Theorem 1.1

A subset of a countable set is countable

**Proof** Let $A$ be a countable set and let $B \subseteq A$

If $A$ or $B$ is finite, then

obviously $B$ is countable

Note

$\mathbb{R}^n$ with usual metric is called an $n$-dimensional Euclidean space.

Example 5.

Consider $\mathbb{R}^n$. Let $p \geq 1$

Proof:

Define $d(x, y) = \left[ \sum_{i=1}^{n} |x_i - y_i|^p \right]^{1/p}$, $x = (x_1, x_2, \ldots, x_n)$,

$y = (y_1, y_2, \ldots, y_n)$ then $d$ is a metric on $\mathbb{R}^n$.

The proof is similar to that of example 4.

Example 6.

Let $x, y \in \mathbb{R}^2$. Then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. We define $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Then $m$ is a metric on $\mathbb{R}^2$.

Proof:

$d(x, y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$

$d(x, y) = 0 \Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0$

$\Leftrightarrow |x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$

$\Leftrightarrow x_1 = y_1$ and $x_2 = y_2$

$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$

$\Leftrightarrow x = y$

$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$

$= |y_1 - x_1| + |y_2 - x_2|$

$= d(y, x)$

Course code - 7BMA5C1

Core - Real Analysis

Unit - I

Countable Sets:

Defn: Bijection

Two Sets A and B are said to be equivalent if There exists a bijection f from A to B.

Note:

* Two finite sets A and B are equivalent iff They have the same number of elements

* A finite Set cannot be equivalent to a proper Subset of itself

* An infinte set can be equivalent to a proper Subset.

Example: 1

Let $A = N$ and $B = \{2, 4, 6, \ldots 2n, \ldots\}$

Then $f : A \to B$ defined by $f(n) = 2n$ is a bijection. Hence A is equivalent to B even Thorugh A has actually more elements Than B.

Hence let A and B be both infinite

Since A is countably infinite we can write $A = \{a_1, a_2, \dots, a_n, \dots\}$. Let $a_{n_1}$ be the first element in A such that $a_{n_1} \in B$. Let $a_{n_2}$ be the first element in A which follows $a_{n_1}$ such that $a_{n_2} \in B$

Proceeding like this we get $B = \{a_{n_1}, a_{n_2}, \dots\}$ Thus all the elements of B can be labelled by using the elements of N. Hence B is countable.

Theorem 1.2

$Q^+$ is countable

Proof: Take all positive rational numbers whose numerator and denominator add up to 2. we have only one number namely $1/1$.

Next we take all positive rational numbers whose numerator & denominator up to 3.

we have $1/2$ and $\frac{2}{1}$

Next we take all positive rational numbers whose numerator & denominator add up to 4.

we have, $\frac{3}{1}$, $\frac{2}{2}$ and $1/3$.

Proceedings like this, we can list all the positive rational number together from the

$\therefore f$ is 1-1

Now, Suppose $(a_m, a_n) \in A \times B$

Then $(m, n) \in N \times N$ and $f(m, n) = (a_m, a_n)$

$\therefore f$ is onto. Hence $f$ is a bijection.

Hence $A \times B$ is equivalent to $N \times N$

which is countable

Hence $A \times B$ is countable

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Theorem 1.6

Let $A$ be a countably infinite set and $f$ be a mapping of $A$ onto a set $B$. Then $B$ is countable.

Proof:

Let $A$ be a countably infinite set and $f : A \to B$ be an onto map.

Let $b \in B$. Since $f$ is onto, there exists at least one pre-image for $b$. Choose one element $a \in A$ such that $f(a) = b$.

Now, define $g : B \to A$ by $g(b) = a$

Clearly $g$ is 1-1

$\therefore B$ is equivalent to a subset of the countable set $A$.

$\therefore B$ is countable.

Example 2

$\mathbb{N}$ is equivalent to $\mathbb{Z}$.

Then function $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \dfrac{1-n}{2} & \text{if } n \text{ is odd} \end{cases}$$

is a bijection. Hence $\mathbb{N}$ is equivalent to $\mathbb{Z}$.

Definition:

A set $A$ is said to be <u>countably infinite</u> if $A$ is equivalent to the set of natural numbers $\mathbb{N}$.

$A$ is said to be <u>countable</u> if it is <u>finite</u> or <u>countably infinite</u>.

Note.

Let $A$ be a countably infinite set. Then there is a bijection $f$ from $\mathbb{N}$ to $A$.

Let $f(1) = a_1$, $f(2) = a_2$, ..... $f(n) = a_n$.

Then $A = \{a_1, a_2, \ldots\ldots a_n, \ldots\}$

Thus all the elements of $A$ can be labelled by using the elements of $\mathbb{N}$.

## Example 9.

Let $p \geq 1$. Let $l_p$ denote the set of all sequences $(x_n)$ such that

$\sum_{1}^{\infty} |x_n|^p$ is convergent. Define $d(x,y) = \left[\sum_{n=1}^{\infty} |x_n - y_n|^p\right]^{1/p}$

where $x = (x_n)$ and $y = (y_n)$

Then $d$ is metric on $l_p$.

*Proof:* Let $a, b \in l_p$

First we prove $d(a,b)$ is a real number.

By Minkowski's inequality we have

$$\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p} \rightarrow (1)$$

Since $a, b \in l_p$ the right hand side of (1) has a finite limit as $n \rightarrow \infty$

$\therefore \left(\sum_{i=1}^{\infty} |a_i + b_i|^p\right)^{1/p}$ is a convergent series

Similarly we can prove that $\left(\sum_{i=1}^{\infty} |a_i - b_i|^p\right)^{1/p}$ is also a convergent series and hence $d(a,b)$ is a real number.

Now, taking limit as $n \rightarrow \infty$ in (1) we get

$$\left(\sum_{i=1}^{\infty} |a_i + b_i|^p\right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |b_i|^p\right)^{1/p} \rightarrow (2)$$

Obviously $d(x,y) \geq 0$.

Problem 3.

1. Any infinite set is equivalent to a proper subset of itself.

Soln: Let $A$ be an infinite set.

By problem 2 above, $A$ contains a countably infinite ~~subset~~ subset $B = \{a_1, a_2, \dots, a_n, \dots\}$

Clearly $A = (A-B) \cup B$

Now consider the following subset $C$ of $A$ given by

$C = (A-B) \cup \{a_2, a_3, \dots, a_n, \dots\} = A - \{a_1\}$

clearly $C$ is a proper subset of $A$

consider the function $f: A \to C$ defined by

$f(x) = x$ if $x \in A - B$ and

$f(a_n) = a_{n+1}$

Obviously $f$ is a bijection. Hence $A$ is equivalent to $C$.

## UNCOUNTABLE SETS

Definition:

A set which is not countable is called uncountable.

All the infinite sets we have considered in the previous section are countable.

We shall now give an example of an uncountable set.

$$= \frac{1}{p} + \frac{1}{q} \quad \left[\text{since } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j\right]$$

$$= 1$$

using this in (1) we get $\sum_{i=1}^{n} |a_i||b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$

$$\therefore \sum_{i=1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$$

Note:

If we put $p = 2 = q$ in Holder's inequality we get the following inequallity which is known as **cauchy** - Schwarz inequality.

$$\sum_{i=1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |b_i|^2\right]^{1/2}$$

Theorem 1.10

(Minkowski's Inequality)

If $p \geq 1$, $\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p}$

where, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers

Proof: This inequality is trivial when $p = 1$, let $p > 1$

$y = (y_1, y_2, \ldots, y_n)$

Then $d$ is a metric on $\mathbb{R}^n$. This is called the usual metric on $\mathbb{R}^n$.

Proof: $d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} \geq 0$

$d(x,y) = 0 \iff \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} = 0$

$\iff (x_i - y_i)^2 = 0$ for all $i = 1, 2, \ldots, n$

$\iff x_i = y_i$ for all $i = 1, 2, \ldots, n$

$\iff (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$

$\iff x = y$

Also $d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

$= \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{1/2}$

$= d(y,x)$

To prove the triangle inequality, take $a_i = x_i - y_i$, $b_i = y_i - z_i$ and $p = 2$ in Minkowski's inequality we get

$$\left[\sum_{i=1}^{n} (x_i - z_i)^2\right]^{1/2} \leq \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2} + \left[\sum_{i=1}^{n} (y_i - z_i)^2\right]^{1/2}$$

$d(x,z) \leq d(x,y) + d(y,z)$

$d$ is a metric on $\mathbb{R}^n$

We have $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$

$\therefore \sup\{|f(x) - h(x)|\} \leq \sup\{|f(x) - g(x)|\} + \sup\{|g(x) - h(x)|\}$

$\therefore d(f, h) \leq d(f, g) + d(g, h)$

Hence $d$ is a metric on $M$.

Example 11.

Let $M$ be the set of all sequences in $\mathbb{R}$.

Let $x, y \in M$ and let $x = (x_n)$ and $y = (y_n)$

Define $d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$

then $d$ is a metric on $M$.

Proof: Let $x, y \in M$. First we prove that $d(x, y)$ is a real number $\geq 0$.

We have $\frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)} \leq \frac{1}{2^n}$ for all $n$

Also, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series

$\therefore \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)}$ is a convergent series

$\therefore d(x, y)$ is a real number and $d(x, y) \geq 0$

Now, $d(x, y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)} = 0$

$\Leftrightarrow |x_n - y_n| = 0$ for all $n$

$\Leftrightarrow x_n = y_n$ for all $n$

$\Leftrightarrow x = y$.

Also, $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|y_n - x_n|)}$

$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n(1+|y_n - x_n|)}$

$= d(y,x)$

Now, let $x, y, z \in M$, then

$$\frac{|x_n - z_n|}{1+|x_n - z_n|} = 1 - \frac{1}{1+|x_n - z_n|} \leq 1 - \frac{1}{1+|x_n - y_n| + |y_n - z_n|}$$

$$= \frac{|x_n - y_n| + |y_n - z_n|}{1+|x_n - y_n| + |y_n - z_n|}$$

$$= \frac{|x_n - y_n|}{1+|x_n - y_n| + |y_n - z_n|} + \frac{|y_n - z_n|}{1+|x_n - y_n| + |y_n - z_n|}$$

$$\leq \frac{|x_n - y_n|}{1+|x_n - y_n|} + \frac{|y_n - z_n|}{1+|y_n - z_n|}$$

Multiplying both sides of this inequality by $1/2^n$ and taking the sum

from $n=1$ to $\infty$ we get $d(x,z) \leq d(x,y) + d(y,z)$

$\therefore$ $d$ is a metric on $M$.

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Example 12.

Let $l^{\infty}$ denote the set of all bounded sequences of real numbers.

Let $x = (x_n)$ and $y = (y_n) \in l^{\infty}$ define $d$ on $l^{\infty}$

as $d'(x,y) = \text{lub}\, |x_n - y_n|$

Then $d$ is a metric on $l^{\infty}$

$$\therefore\ x^{1/p}\, y^{(1-1/p)} - \frac{x}{p} - \frac{y}{q} \leq 0 \quad \left(\text{since } 1 - \frac{1}{p} = \frac{1}{q}\right)$$

$$\therefore\ x^{1/p}\, y^{1/q} \leq \frac{x}{p} + \frac{y}{q}$$

Now to prove Holder's inequality, we apply the above inequality to the

numbers $x_j = \dfrac{|a_j|^p}{\sum_{i=1}^{n} |a_i|^p}$ ; $y_j = \dfrac{|b_j|^q}{\sum_{i=1}^{n} |b_i|^q}$ for each $j = 1, 2, \ldots n$

We get

$$\frac{|a_j|\,|b_j|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}} \leq \frac{x_j}{p} + \frac{y_j}{q}$$

for all $j = 1, 2, \ldots n$

Adding these $n$ inequalities we get

$$\frac{\sum_{i=1}^{n} |a_i|\,|b_i|}{\left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}} \leq \sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right)$$

Now $\displaystyle\sum_{j=1}^{n} \left(\frac{x_j}{p} + \frac{y_j}{q}\right) = \frac{1}{p} \sum_{j=1}^{n} x_j + \frac{1}{q} \sum_{j=1}^{n} y_j$

[illegible]

[illegible] $|x_1 - z_1| + |x_2 - z_2|$ [illegible]

$= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$

$\le |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$

$= \{|x_1 - y_1| + |x_2 - y_2|\} + \{|y_1 - z_1| + |y_2 - z_2|\}$

[illegible]

Hence $d$ is a metric on [illegible]

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Note:

[illegible] in $\mathbb{R}^n$ we define

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ where [illegible]

[illegible]

[illegible]

[illegible]

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Example 7

In $\mathbb{R}^n$ we define

$d(x,y) = \max\{|x_i - y_i|$ : $i = 1, 2, \ldots$ [illegible]

$x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

Then $d$ is a metric on $\mathbb{R}^n$

Proof: $d(x,y) = \max\{|x_i - y_i|$ [illegible]

[illegible] $\max\{|x_i - y_i|$ [illegible]

[illegible] for $i = 1, 2, \ldots, n$

Theorem 1.7

Countable union of countable sets is countable

Proof: Let $A_1, A_2, \ldots, A_n, \ldots$ be a countable family of countable sets.

Case (i) Let each $A_i$ be countably infinite.

Let $A_1 = \{a_{11}, a_{12}, \ldots, a_{1n}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, \ldots, a_{2n}, \ldots\}$

$\ldots\ldots\ldots$

$A_n = \{a_{n1}, a_{n2}, \ldots, a_{nn}, \ldots\}$

$\ldots\ldots\ldots$

Now we define a map [illegible]

Clearly $f$ is 1-1 onto

Also by theorem 1.5 $N \times N$ is countably infinite

Hence by theorem 1.6, $\cup A_n$ is countably infinite

Case (ii) Let each $A_i$ be countable

For each $i$ chose a set $B_i$ such that $B_i$ is a countably infinite set and $A_i \subseteq B_i$.

Then $\cup A_i \subseteq \cup B_i$

Now, $\cup B_i$ is countable (by case (i))

$\therefore \cup A_i$ is countable (by Theorem 1.1)

Chapter 2

## METRIC SPACES

### Definition and Examples

**Definition:**

A metric space is a non-empty set M together with a function $d: M \times M \to \mathbb{R}$ satisfying the following conditions

(i) $d(x,y) \geq 0$ for all $x, y \in M$

(ii) $d(x,y) = 0$ iff $x = y$

(iii) $d(x,y) = d(y,x)$ for all $x, y \in M$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in M$

d is called a metric or distance function and $d(x,y)$ is called the distance between x and y.

**Note:**

The metric space M with the metric d is denoted by (M, d) or simply by M when the underlying metric d is clear from the context.

**Example 1.**

In R we define $d(x,y) = |x-y|$ then d is a metric on R. This is called the usual metric on R.

**Proof:** Clearly $d(x,y) = |x-y| \geq 0$

Also $d(x,y) = 0 \Leftrightarrow |x-y| = 0$

$\Leftrightarrow x = y$

then

$$\sum_{i=1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots b_n$ are real numbers.

**Proof:** First we shall prove the inequality

$$x^{1/p} y^{1/q} \le \frac{x}{p} + \frac{y}{q} \text{ where } x \ge 0 \text{ and } y \ge 0$$

this inequality is trivial if $x = 0$, or $y = 0$

Now, let $x, y > 0$

Consider $f(t) = t^{\lambda} - \lambda t + \lambda - 1$ where $\lambda = \frac{1}{p}$ and $t \ge 0$

then $f'(t) = \lambda t^{\lambda - 1} - \lambda = \lambda(t^{\lambda - 1} - 1)$

$\therefore f(1) = f'(1) = 0$

Also $f'(t) > 0$ for $0 < t < 1$ and $f'(t) < 0$ for $t > 1$

$\therefore f(t) \le 0$ for all $t \ge 0$ and and in particular $f\left(\frac{x}{y}\right) \le 0$

$$\therefore \left(\frac{x}{y}\right)^{\lambda} - \lambda\left(\frac{x}{y}\right) + \lambda - 1 \le 0$$

$$\therefore \left(\frac{x}{y}\right)^{1/p} - \frac{1}{p}\left(\frac{x}{y}\right) + \frac{1}{p} - 1 \le 0$$

Multiplying by $y$ we get $x^{1/p} y^{(1 - 1/p)} - \frac{x}{p} - \left(1 - \frac{1}{p}\right)y \le 0$

$d(x,y) = |x-y|$

$= |y-x|$

$= d(y,x)$

Now, let $x, y, z \in R$

Then $d(x,z) = |x-z| = |x-y+y-z|$

$\leq |x-y| + |y-z|$

$= d(x,y) + d(y,z)$

$d(x,z) \leq d(x,y) + d(y,z)$

Hence $d$ is a metric on $R$.

Note:

In what follows whenever we consider $R$ as a metric space the under lying metric is taken to be the usual metric unless otherwise stated

Example 2:

In $C$ we define $d(z,w) = |z-w|$. Then $d$ is a metric on This is called the usual metric on $C$.

Note:

If the complex number $z = x+iy$ is ~~ale~~ identified with the point $(x,y)$ the two dimensional Euclidean plane then the above distance formula take the form

$d(z,w) = \sqrt{(x-u)^2+(y-v)^2}$ where $z = x+iy$ and $w = u+iv$

this is nothing but the usual distance between the points $(x,y)$ and $(u,v)$ in the plane.

Example 3.

on any non-empty set $M$ we define $d$ as as follows $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x = y \end{cases}$

Problem 6.

Let $(M, d)$ be a metric space. Define $d_1(x,y) = \min\{1, d(x,y)\}$. Prove that $d_1$ is a metric

Sol: $d_1(x,y) = \min\{1, d(x,y)\} \geq 0$

$\therefore d_1(x,y) \geq 0$

$$d_1(x,y) = 0 \iff \min\{1, d(x,y)\} = 0$$
$$\iff d(x,y) = 0$$
$$\iff x = y$$

Also $d_1(x,y) = \min\{1, d(x,y)\}$
$$= \min\{1, d(y,x)\}$$
$$= d_1(y,x)$$

Now, let $x, y, z \in M$

Then $d_1(x,z) = \min\{1, d(x,z)\} \leq 1$

To prove $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$

If $d_1(x,y) = 1$ or $d_1(y,z) = 1$ the inequality is obvious

Let $d_1(x,y) < 1$ and $d_1(y,z) < 1$. Then

$$d_1(x,y) + d_1(y,z) = \min\{1, d(x,y)\} + \min\{1, d(y,z)\}$$
$$= d(x,y) + d(y,z)$$
$$\geq d(x,z)$$
$$\geq \min\{1, d(x,z)\}$$
$$= d_1(x,z)$$

Thus $d_1(x,y) + d_1(y,z) \geq d_1(x,z)$

$\therefore d_1$ is a metric on $M$.

Problem 7.

Let $M$ be a non empty set. Let $d : M \times M \to R$ be a function such that

(i) $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in M$. Prove that $d$ is metric on $M$.

Soln: put $y = x$ in (ii)

we have $d(x,x) \leq d(x,z) + d(x,z)$

$\therefore 0 \leq 2d(x,z)$ by (i)

$\therefore d(x,z) \geq 0$

Now, to prove $d(x,y) = d(y,x)$

Putting $z = x$ in (ii) we get $d(x,y) \leq d(x,x) + d(y,x)$

(ie) $d(x,y) \leq d(y,x)$ [using (i)]

Since this is true for all $x, y \in M$ we have $d(y,x) \leq d(x,y)$.

Hence $d(x,y) = d(y,x)$

Now (ii) can be written as $d(x,y) \leq d(x,z) + d(z,y)$ which is the triangle inequality

$\therefore d$ is a metric on $M$

Now $d(x,y) = d_1(x,y) + d_2(x,y)$

$= d_1(y,x) + d_2(y,x)$

$= d(y,x)$

Let $x, y, z \in M$. Then we have

$d_1(x,z) \leq d_1(x,y) + d_1(y,z)$ and

$d_2(x,z) \leq d_2(x,y) + d_2(y,z)$

Adding we get $d(x,z) \leq d(x,y) + d(y,z)$

$\therefore$ $d$ is a metric on $M$.

Problem 2.

Determine whether $d(x,y)$ defined on $R$ by $d(x,y) = (x-y)$ is a metric or not.

Soln: Let $x, y \in R$

$d(x,y) = (x-y)^2 \geq 0$

$d(x,y) = (x-y)^2 = (y-x)^2$

$= d(y,x)$

But triangle inequality does not hold

Take $x = -5$, $y = -4$ and $z = 4$

Then $d(x,y) = (-5+4)^2 = 1$

$d(y,z) = (-4-4)^2 = 64$

$d(x,z) = (4+5)^2 = 81$

Hence $d(x,z) > d(x,y) + d(y,z)$

Hence triangle inequality does not hold

$\therefore$ $d$ is not a metric on $R$.

beginning omitting those which are already l[illegible]

Thus we obtain the set $\{1, \frac{1}{2}, 2, 3, [illegible]\}$ [illegible]

This set contains every positive rational number, each occurring exactly once. Thus $Q^+$ is countable.

---

Theorem : 3

$Q$ is countable

Proof

We know that $Q^+$ is countable

Let $Q^+ = \{r_1, r_2, \dots, r_n, \dots\}$

$\therefore Q = \{0, \pm r_1, \pm r_2, \dots \pm r_n, \dots\}$

Let $f : N \to Q$ is be defined by

$f(1) = 0$, $f(2n) = r_n$ and $f(2n+1) = -r_n$

Clearly $f$ is a bijection and hence $Q$ is countable

---

Theorem : 4

$N \times N$ is countable

Proof : $N \times N = \{(a, b) / a, b \in N\}$

Take all ordered pairs $(a, b)$ such that $a+b = 2$

There is only one such pair namely $(1, 1)$

Next take all ordered pairs $(a, b)$ such that $a+b = 3$

Theorem 1.8

$(0,1]$ is uncountable.

Proof: Every real number in $(0,1]$ can be written uniquely as a non-terminating decimal $.a_1a_2 \ldots a_n \ldots$ where $0 \le a_i \le 9$ for each $i$ subject to the following restriction that any terminating decimal $.a_1a_2 \ldots a_n 000 \ldots$ is written as $.a_1a_2a_3 \ldots (a_n - 1)999 \ldots$

For example, $.54 = .53999 \ldots$

$1 = .999 \ldots$

Suppose $(0,1]$ is countable.

Then the elements of $(0,1]$ can be listed as $\{x_1, x_2, \ldots x_n, \ldots\}$ where $x_1 = .a_{11} a_{12} \ldots a_{1n} \ldots$

$x_2 = .a_{21} a_{22} \ldots a_{2n} \ldots$

$\ldots$

$\ldots$

$x_n = .a_{n1} a_{n2} \ldots a_{nn}$

$\ldots$

$\ldots$

Now, for each positive integer $n$ choose an integer $b_n$ such that $0 \le b_n \le 9$ and $b_n \ne 0$ and $b_n \ne$

Let $y = .b_1 b_2 b_3 \ldots$

Clearly $y \in (0,1]$

Also $y$ is different from each $x_i$ at least in the $i^{th}$ place

Hence $y \ne x_i$ for each $i$ which is a contradiction

Hence $(0,1]$ is uncountable

where have (1, 2) and (2, 1)

Next take all ordered pairs $(a, b)$ such that $a+b=4$

we have (3, 1), (2, 2) and (1, 3)

Proceedings like this and listing all the ordered pairs together from the beginning, we get the set $\{(1,1),(1,2),(2,1),(3,1),(2,2),(1,3),\dots\}$

This set contains every ordered pair belonging to $N \times N$ exactly once.

Thus $N \times N$ is countable

Theorem 1.5

If A and B are countable sets then $A \times B$ is also countable.

Proof: We assume that A and B are countably infinite

Let $A = \{a_1, a_2, \dots, a_n, \dots\}$, $B = \{b_1, b_2, \dots, b_n, \dots\}$

Now define $f : N \times N \to A \times B$ by $f(i, j) = (a_i, b_j)$

We claim that $f$ is a bijection

Suppose $x = (p, q) \in N \times N$ and $y = (u, v) \in N \times N$

Now, $f(x) = f(y) \Rightarrow (a_p, b_q) = (a_u, b_v)$

$\Rightarrow a_p = a_u,\ b_q = b_v$

$\Rightarrow p = u$ and $q = v$

$\Rightarrow (p, q) = (u, v)$

$\Rightarrow x = y$

Problem 9 :-

In a metric space $(M, d)$ prove that

$|d(x,z) - d(y,z)| \leq d(x,y)$ for all $x, y, z \in M$

Soln: Let $x, y, z \in M$

We have $d(x,z) \leq d(x,y) + d(y,z)$

$\therefore d(x,z) - d(y,z) \leq d(x,y) \rightarrow (1)$

Interchanging $x$ and $y$ in (1) we get

$d(y,z) - d(x,z) \leq d(y,x)$

$= d(x,y)$

$\therefore d(y,z) - d(x,z) \leq d(x,y) \rightarrow (2)$

From (1) and (2) we get $|d(x,z) - d(y,z)| \leq d(x,y)$

Problem 8.

If $(M_1, d_1), (M_2, d_2), \ldots\ldots, (M_n, d_n)$ are metric spaces then $M_1 \times M_2 \times \ldots \times M_n$ is a metric space with metric $d$ defined by

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \text{ where } x = (x_1, x_2, \ldots, x_n),$$

$$y = (y_1, y_2, \ldots, y_n)$$

Soln: $d(x, y) \sum_{i=1}^{n} d_i(x_i, y_i) \geq 0$

Also $d(x, y) = 0 \Leftrightarrow \sum_{i=1}^{n} d_i(x_i, y_i) = 0$

$\Leftrightarrow d_i(x_i, y_i) = 0$ for all $i = 1, 2, \ldots n$

$\Leftrightarrow x_i = y_i$ for all $i = 1, 2, \ldots, n$

$\Leftrightarrow (x_1, \ldots x_n) = (y_1, \ldots y_n)$

$\Leftrightarrow x = y$

Also $d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i)$

$= \sum_{i=1}^{n} d_i(y_i, x_i)$

$= d(y, x)$

Now, let $x, y, z \in M$

Then $d(x, z) = \sum_{i=1}^{n} d_i(x_i, z_i)$

$\leq \sum_{i=1}^{n} [d_i(x_i, y_i) + d_i(y_i, z_i)]$

$= \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(y_i, z_i)$

$= d(x, y) + d(y, z)$

$\therefore d(x, z) \leq d(x, y) + d(y, z)$

Hence $d$ is a metric $M$.

Problem 1.

Any countably infinite set is equivalent to a proper subset of itself.

Sol: Let A be a countably infinite set.

Hence $A = \{a_1, a_2, \ldots, a_n, \ldots\}$

Let $B = \{a_2, a_3, \ldots, a_n, \ldots\}$

Clearly B is a proper subset of A.

Define a map $f: A \to B$ by $f(a_n) = a_{n+1}$

Clearly f is a bijection. Hence A is equivalent to B.

---

Problem 2:

Any infinite set contains a countably infinite subset.

Sol: Let A be an infinite set.

Choose any element $a_1 \in A$

Now, since A is infinite set, we can choose another element $a_2 \in A - \{a_1\}$.

Now, suppose we have choosen $a_1, a_2, \ldots, a_n$ from A.

$\therefore$ Since A is infinite, $A - \{a_1, a_2, \ldots, a_n\}$ is also infinite.

$\therefore$ We can choose $a_{n+1}$ from $A - \{a_1, a_2, \ldots, a_n\}$

Now, $B = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$ is countably infinit subset of A.

Clearly, $\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1/p} \rightarrow ①$

Now $\sum_{i=1}^{n} [|a_i| + |b_i|]^p = \sum_{i=1}^{n} [|a_i| + |b_i|]^{p-1} (|a_i| + |b_i|)$

$= \sum_{i=1}^{n} |a_i| (|a_i| + |b_i|)^{p-1} + \sum_{i=1}^{n} |b_i| (|a_i| + |b_i|)^{p-1}$

$\leq \left[\sum_{i=n} |a_i|^p\right]^{1/p} \left[\sum_{i=1} (|a_i| + |b_i|)^{(p-1)q}\right]^{1/q} +$

$\left[\sum_{i=n} |b_i|^p\right]^{1/p} \left[\sum_{i=1} (|a_i| + |b_i|)^{(p-1)q}\right]^{1/q}$

where $\frac{1}{p} + \frac{1}{q} = 1$ (using Holder's inequality)

Now, Since $\frac{1}{p} + \frac{1}{q} = 1$ we have $p + q = pq$

Hence $(p-1)q = p$

$\therefore$ Dividing by $\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1/q}$ we get

$\left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1-\frac{1}{q}} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|\right]^{1/p}$

$\therefore \left[\sum_{i=1}^{n} (|a_i| + |b_i|)^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p}$

$\Leftrightarrow x_i = y_i$ for all $i = 1, 2, \dots n$

$\Leftrightarrow (x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$

$\Leftrightarrow x = y$

Also, $d(x, y) = \max\{|x_i - y_i|\}$

$= \max\{|y_i - x_i|\}$

$= d(y, x)$

Now, let $x, y, z \in R^n$. Since each $x_i, y_i, z_i \in R$

we have, $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for all $i = 1, 2$

$\therefore \max|x_i - z_i| \leq \max|x_i - y_i| + \max|y_i - z_i|$

$\therefore d(x, z) \leq d(x, y) + d(y, z)$

Hence $d$ is a metric on $R^n$

---

Example 8.

Let $c_1, c_2, \dots c_n$ be given fixed positive real numbers.

Proof:

Let $x, y \in R^n$ where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$

We define $d(x, y) = \sum_{i=1}^{n} c_i |x_i - y_i|$

then $d$ is an metric on $R^n$.

---

Note:

A non-empty set $M$ can be provided with different metrices.

For example, $R^n$ has been provided with five different metrices as seen from the example 4 to 8.

Soln:

$d(x, y) = lub\,|x_n - y_n| \geq 0$

$d(x, y) = 0 \Leftrightarrow lub\,|x_n - y_n| = 0$

$\Leftrightarrow |x_n - y_n| = 0$ for all $1 \leq n < \infty$

$\Leftrightarrow x_n = y_n$ for $1 \leq n < \infty$

$\Leftrightarrow (x_n) = (y_n)$

$\Leftrightarrow x = y$

Now, $d(x, y) = lub\,|x_n - y_n|$

$= lub\,|y_n - x_n|$

$= d(y, x)$

Let $z = (z_n)$.

Now, $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$

$\leq lub\,|x_n - y_n| + lub\,|y_n - z_n|$

$= d(x, y) + d(y, z)$

$\therefore lub\,|x_n - z_n| \leq d(x, y) + d(y, z)$

$d(x, z) \leq d(x, y) + d(y, z)$

$\therefore d$ is a metric on $l^{\infty}$.

Solved Problems:-

Problem 1.

Let $d_1$ and $d_2$ be two metrics on M. Define $d(x, y) = d_1(x, y) + d_2(x, y)$. Prove that $d$ is a metric M.

Soln: $d(x, y) = d_1(x, y) + d_2(x, y) \geq 0$

$d(x, y) = 0 \Leftrightarrow d_1(x, y) + d_2(x, y) = 0$

$\Leftrightarrow d_1(x, y) = 0$ and $d_2(x, y) = 0$

$\Leftrightarrow x = y$

Example: 1 $\{2, 4, 6, \ldots, 2n, \ldots\}$ is a Countable set

Example: 2

$\mathbb{Z}$ is countable

Example: 3

Let $A = \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}$

The function $f: \mathbb{N} \to A$ defined by $f(n) = \frac{n}{n+1}$ is a bijection. Hence $A$ is Countable.

Theorem 1.1

A subset of a countable set is countable.

Proof: Let $A$ be a countable set and let $B \subseteq A$. If $A$ or $B$ is finite then obviously $B$ is Countable. Hence let $A$ and $B$ be both infinite.

Since $A$ is countably infinite we can write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let $a_{n_1}$ be the first element in $A$ such that $a_{n_1} \in B$. Let $a_{n_2}$ be the first element in $A$ which follows $a_{n_1}$ such that $a_{n_2} \in B$.

Proceedings like this we get $B = \{a_{n_1}, a_{n_2}, \ldots\}$. thus all the elements of $B$ can be labelled by using the elements of $\mathbb{N}$

Hence $B$ is countable.

Then d is a metric on M. This is called the discreke metric on M.

Proof:

Clearly $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$

Also $d(x,y) = d(y,x) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$\therefore d(x,y) = d(y,x)$ for all $x, y \in M$

Now let $x, y, z \in M$

case (i) $x = z$

Then $d(x,z) = 0$

Also, $d(x,y) + d(y,z) \geq 0$

$\therefore d(x,z) \leq d(x,y) + d(y,z)$

case (ii) $x \neq z$

Then $d(x,z) = 1$

Also, since $x, z$ are distinct, $y$ can not be equal to both $x$ and $z$.

Hence either $y \neq x$ or $y \neq z$.

$\therefore d(x,y) + d(y,z) \geq 1$

$d(x,z) \leq d(x,y) + d(y,z)$

Thus $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in M$

Hence d is a metric on M.

Example 4.

In $R^n$ we define $d(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2}$

where $x = (x_1, x_2, \ldots, x_n)$ and

$$d_1(x,y) = 0 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} = 0$$

$$\Leftrightarrow d(x,y) = 0$$

$$\Leftrightarrow x = y \quad (\text{since } d \text{ is a metric})$$

Also, $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

$$= \frac{d(y,x)}{1+d(y,x)}$$

$$= d_1(y,x)$$

Now let $x, y, z \in M$

then $d_1(x,z) = \dfrac{d(x,z)}{1+d(x,z)}$

$$= 1 - \frac{1}{1+d(x,z)}$$

$$\leq 1 - \left[\frac{1}{1+d(x,y)+d(y,z)}\right]$$

$$= \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\leq \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$$= d_1(x,y) + d_1(y,z)$$

Thus $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$

$\therefore d_1$ is a metric on $M$.

Corollary 1.

Any subset A of R which contain [0, 1] is the uncountable.

Proof: Suppose A is countable.

By theorem 1.1 any subset of A is countable. Hence we get [0, 1] is countable which is a contradiction.

$\therefore$ A is uncountable.

Corollary 2: R is uncountable

The results follows directly by taking A = R

Corollary 3:

The set S of irrational numbers is uncountable.

Proof: Suppose S is countable

we know that Q is countable

$\therefore$ $S \cup Q = R$ is countable which is contradiction.

$\therefore$ S is uncountable.

(1)

Inequalities of Holder and Minkowski

Theorem 1.9

(Holder's Inequality) If $p > 1$ and $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$

We can choose, $x_n \in B(x, 1/n) \cap A$

Such that,

$x_n \neq x_1, x_2, \ldots x_{n-1}$ for each $n$

$\therefore (x_n)$ is a Sequence of points in A

Also, $d(x_n, x) < 1/n \ \forall n$

$\therefore \lim_{n \to \infty} d(x_n, x) = 0$

$\therefore (x_n) \to x$

conversely,

Suppose, $\exists$ a Sequence $(x_n)$ in $A \ni : (x_n) \to x$

Then, for any $r > 0$ there exists a +ve Integer $n_0$ Such that $d(x_n, x) < r \ \forall n \geq n_0$

$\therefore x_n \in B(x, r) \ \forall n \geq n_0$

$B(x, r) \cap A \neq \phi$

$\therefore \boxed{x \in \bar{A}}$ ( $\because$ If $x \in \bar{A}$ iff $B(x, r) \cap A \neq \phi \ \forall r > 0$)

Further, if $(x_n)$ is a Sequence of distinct points in A is infinite

$\therefore x \in D(A)$

$\therefore x$ is a limit point of A.

Complete:

A Metric Space M is Said to be Complete if every cauchy Sequence in M converges to a points in M.

Theorem: 2

Any discrete Metric Space is Complete.

Proof: Let $(M, d)$ be a discrete Metric Space

Let $(x_n)$ be a Cauchy Sequence in M

Then $\exists$ a +ve integer $n_0 \ni$:

$d(x_n, x_m) < 1/2 \ \forall n, m \geq n_0$

Thus, $(x_n)$ is a Sequence in A converging to $x$.

$$\boxed{x \in \bar{A}}$$

Since. A is closed. iff $A = \bar{A}$

$$\therefore \boxed{x \in A}$$

Thus. every cauchy Sequence $(x_n)$ in A converges to a point in A.

$\therefore$ A is Complete.

u.R

Cantor Intersection Theorem:-

Let M be a metric Space M is complete iff for every Sequence $(F_n)$ of non-empty closed Subset of M Such that $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$ and $(d(F_n)) \to 0$

$$\bigcap_{n=1}^{\infty} F_n \text{ is non-empty.}$$

Let M be a metric Space let $(F_n)$ be a Sequence of closed Subsets of M Such that

$F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$ and $(d(F_n)) \to 0$

T.P:

$$\bigcap_{n=1}^{\infty} F_n \neq \phi$$

for each the integer n

choose a point $x_n \in F_n$

$\Rightarrow x_n, x_{n+1}, x_{n+2}, \dots$ all lies in $F_n$.

$\Rightarrow x_m \in F_n \ \forall \ m \geq n$

Since, $(d(F_n)) \to 0$, given $\epsilon > 0$. $\exists$ a +ve Integer $n_0$ Such that

$d(F_n) < \epsilon, \ \forall \ n \geq n_0$

$\Rightarrow d(F_{n_0}) < \epsilon$

$\Rightarrow d(x, y) < \epsilon \ \ \forall \ x, y \in F_n$

$\Rightarrow d(x,y) < r_1$

$\Rightarrow d(x,y) < d(a,x) - r$

$\Rightarrow d(a,x) > d(x,y) + r$

Now,

$d(a,x) \le d(a,y) + d(y,x)$

$\therefore d(a,y) \ge d(a,x) - d(y,x)$

$d(a,y) > d(x,y) + r - d(y,x)$

$\Rightarrow d(a,y) > r$

$\Rightarrow y \notin B[a,r]$

$y \in B[a,r]^c$

Hence. $B(x,r_1) \subseteq B[a,r]^c$

$\therefore B[a,r]^c$ is open in M

(i.e) $B[a,r]$ is closed.

Hence every closed ball is a closed set.

Thm

2) In any metric space arbitrary intersection of closed sets in closed.

Let $(M,d)$ be a metric space

Let $\{A_i \mid i \in I\}$ be a collection of closed set

To prove:

$\bigcap A_i$ is closed

$i \in I \Rightarrow$ (i.e) to prove that $\bigcap_{i \in I} A_i^c$ is open

using De-morgan's law.

$$\boxed{\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c}$$

Since, each $A_i$ is closed,

then $A_i^c$ is open.

$\therefore (x_n) \to x$ and $(y_n) \to y$

Also, $(x_n)$ is a Sequence in A and $(y_n)$ is a Sequence in B.

$\therefore x \in \overline{A}$ and $y \in \overline{B}$

$\therefore (x,y) \in \overline{A} \times \overline{B}$

$\therefore \overline{A \times B} \subseteq \overline{A} \times \overline{B}$ — ①

Now, let $(x,y) \in \overline{A} \times \overline{B}$

$\therefore x \in \overline{A}$ and $y \in \overline{B}$

$\therefore \exists$ a Sequence $(x_n)$ in A and a Sequence $(y_n)$ in B

$\ni: (x_n) \to x$ and $(y_n) \to y$.

$\therefore ((x_n, y_n))$ is a Sequence in $A \times B$ Which Converges to $(x,y)$

ie) $((x_n, y_n)) \to (x,y)$

$\therefore (x,y) \in \overline{A \times B}$

$\therefore \overline{A} \times \overline{B} \subseteq \overline{A \times B}$ — (2)

From (1) & (2) We get $\overline{A \times B} = \overline{A} \times \overline{B}$

Hence proved //

2) P.T any non-empty open interval $(a,b)$ in R is of Second category.

Let $(a,b)$ be a non-empty open interval in R.

Suppose, $(a,b)$ is of first category.

Now, $[a,b] = (a,b) \cup \{a\} \cup \{b\}$

$\therefore [a,b]$ is of first Category.

Since, M is complete by cantors intersection theorem,

∃ a point 'x' in M ∋: $x \in \bigcap_{n=1}^{\infty} F_n$

Also, each $F_n$ is disjoint from $A_n$

Hence $x \notin A_n$ for all n.

$\therefore x \notin \bigcup_{n=1}^{\infty} A_n$

$\therefore \bigcup_{n=1}^{\infty} A_n \neq M$

Hence, M is of Second category.

**Definition:**

* A Subset A of a Metric Space M is said to be nowhere dense in M if $\boxed{Int\,\overline{A} = \phi}$

* A Subset A of a metric Space M is Said to be of first category in M if A can be expressed as a Countable union of nowhere dense Sets.

* A Set which is not of first category is Said to be of Second Category.

**Note:**

If A is of first category then $A = \bigcup_{n=1}^{\infty} E_n$, Where $E_n$ is nowhere dense Subsets in M.

**Problem: 1**

Let A, B be a Subsets of R p.T $\overline{A \times B} = \overline{A} \times \overline{B}$

**Proof:** Let $(x,y) \in \overline{A \times B}$

$\therefore$ ∃ a Sequence $((x_n, y_n)) \subset A \times B$ ∋:

$((x_n, y_n)) \to (x,y)$

Note:-

* Let $M$ be a Metric Space and $A \subseteq M$.

Then, $\bar{A} = A \cup D(A)$

* $A$ is closed $\Leftrightarrow$ $A$ contains all its limit points

ie) $A$ is closed $\Leftrightarrow$ $D(A) \subseteq A$

* $x \in \bar{A} \Leftrightarrow B(x,r) \cap A \neq \phi \;\forall r > 0$

Dense:

A subset $A$ of a Metric Space $M$ is said to be dense in $M$ (or) everywhere dense. If $\bar{A} = M$

A Metric Space $M$ is said to be Separable if there exists a countable dense subset in $M$.

Convergent:

Let $(M, d)$ be a Metric Space.

Let $(x_n) = x_1, x_2, \ldots, x_n$ be a Sequence of points in $M$. Let $x \in M$. We say that $(x_n)$ converges to '$x$'. If given $\epsilon > 0$ $\exists$ a +ve integer $n_0$ such that $d(x_n, x) < \epsilon \;\forall n \geq n_0$. Also, '$x$' is called a limit of $(x_n)$.

If $(x_n)$ converges to $x$ we write,

$$\lim_{x \to \infty} x_n = x \quad (\text{or}) \quad (x_n) \to x$$

U.Q

Theorem:

For a Convergent Sequence $(x_n)$ the limit is Unique.

Pf:

Suppose, $(x_n) \to x$ and $(y_n) \to y$

Let $\epsilon > 0$ be given. Then $\exists$ a +ve integer $n_1$ and $n_2$ Such that,

$d(x_n, x) < \frac{1}{2}\varepsilon \ \forall \ n \geq n_0$

$\therefore d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$

$\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \ \forall \ n, m \geq n_0$

Thus, $d(x_n, x_m) < \varepsilon \ \forall \ m, n \geq n_0$

Hence, $(x_n)$ is cauchy Sequence in $M$

**Note:**

The converse of the above theorem is not true.

**Note:** (1) Let $M$ be a metric Space and $A \subseteq M$. Then $\bar{A} = A \cup D(A)$

(2) For any subset $A$ of a metric Space, $d(A) = d(\bar{A})$

Where $d(A)$ is a diameter of $A$.

**Theorem:**

Let $M$ be a metric space and $A \subseteq M$. Then,

(i) $x \in \bar{A}$ iff there exists a Sequence $(x_n)$ in $A$ Such that $(x_n) \to x$

(ii) $x$ is a limit point of $A$ iff $\exists$ a Sequence $(x_n)$ of distinct points in $A$ $\ni$: $(x_n) \to x$.

**Proof:** (i) Let $x \in \bar{A}$

T.P: $(x_n) \to x$

Given, $x \in \bar{A}$, Then $x \in A \cup D(A)$

$\therefore x \in A$ (or) $x \in D(A)$

If $x \in A$, then the constant Sequence $x, x, x, \ldots$ is a Sequence in $A$ converging to $x$.

If $x \in D(A)$ then the open Ball $B(x, 1/n)$ contains infinite number of points of $A$.

# UNIT-II

**Subspace:**

Let $(M,d)$ be a metric Space. Let $M_1$ be a two non-empty Subset of $M$. Then $M_1$ is also a metric Space with the Same metric $d$. We Say that $(M_1,d)$ is a Subspace of $(M,d)$

**Interior of a Set:**

Let $(M,d)$ be a metric Space. Let $A \subseteq M$. Let $x \in A$. Then $x$ is Said to be an interior point of $A$. if there exists a +ve real number $r$ Such that $B(x,r) \subseteq A$

The Set of all Interior points of $A$ is called the interior of $A$ and it is denoted by Int $A$.

**Note:** Int $A \subseteq A$

**closed Sets**

Let $(M,d)$ be a metric Space. Let $A \subseteq M$. Then $A$ is Said to be closed in $M$. If the complement of $A$ is open in $M$

u.th 2m

**Example 1:** In $\mathbb{R}$ with usual metric any closed Interval $[a,b]$ is closed set.

$[a,b]^c = \mathbb{R} - [a,b] = (-\infty, a) \cup (b, \infty)$

Also, $(-\infty, a)$ and $(b,\infty)$ are open in $\mathbb{R}$

i.e. $[a,b]^c$ is open in $\mathbb{R}$.

$\therefore$ $[a,b]$ is closed in $\mathbb{R}$.

2m.x

**Example: 4** P.T $\mathbb{Z}$ is closed.

$$\mathbb{Z}^c = \bigcup_{n=-\infty}^{\infty} (n, n+1)$$

The open Interval $(n,n+1)$ is open and union of open set is open

$\mathbb{Z}^c$ is open. Hence $\mathbb{Z}$ is closed.

$F_1 \supseteq F_2 \supseteq \cdots \cdots \supseteq F_n \supseteq \cdots$

$\overline{F_1} \supseteq \overline{F_2} \supseteq \cdots \cdots \supseteq \overline{F_n} \supseteq \cdots$

clearly, $(\overline{F_n})$ is a decreasing Sequence of closed Sets

Since $(x_n)$ is a cauchy sequence

given $\epsilon > 0$, $\exists$ a +ve integer $n_0$ $\ni$:

$$d(x_m, x_n) < \epsilon, \; \forall \; n, m \geq n_0$$

For any integer $n \geq n_0$

$$d(F_n) = d(\overline{F_n}) \quad (\because d(A) = d(\overline{A}))$$

$$d(\overline{F_n}) < \epsilon \; \forall \; n \geq n_0$$

$$(d(\overline{F_n})) \to 0$$

Hence,

$$\bigcap_{n=1}^{\infty} \overline{F_n} \neq \phi$$

let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$

Then $x, x_n \in \overline{F_n}$

$$d(x, x_n) \leq d(\overline{F_n})$$

$$d(x, x_n) < \epsilon \; \forall \; n \geq n_0$$

$$\therefore (x_n) \to x$$

Hence, $M$ is Complete

Theorem: 4

**Baire's category Theorem**

Any complete Metric Space is of Second category.

But, $[a,b]$ is a Complete Metric Space and hence is of Second category Which is a $\Rightarrow\Leftarrow$

$\therefore$ $(a,b)$ is of Second category.

$d(x_n, x) < \frac{1}{2}\epsilon \quad \forall\ n \geq n_1$ and

$d(x_n, y) < \frac{1}{2}\epsilon \quad \forall\ n \geq n_2$

Let 'm' be a +ve integer such that $m \geq n_1, n_2$

Then, $d(x,y) \leq d(x, x_m) + d(x_m, y)$

$< \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$

$d(x,y) \leq \epsilon$

$\therefore d(x,y) < \epsilon$

Since, $\epsilon > 0$ is arbitrary $d(x,y) = 0$

$\Rightarrow \boxed{x = y}$

Hence, for a convergent sequence $(x_n)$ the limit is unique.

Cauchy Sequence:

Let $(M,d)$ be a metric space. Let $(x_n)$ be a sequence of points of M. $(x_n)$ is said to be a cauchy sequence in M if given $\epsilon > 0$ $\exists$ a +ve integer $n_0 \ni : d(x_m, x_n) < \epsilon \quad \forall\ m, n \geq n_0$

Theorem: Int

Let $(M,d)$ be Metric Space. Then any convergent sequence in M is a cauchy sequence.

Proof:

Let $(x_n)$ be a convergent sequence in M converging to $x \in M$.

ie) $(x_n) \to x$.

Let $\epsilon > 0$ be given.

Then, $\exists$ a +ve integer $n_0 \ni :$

By thm:

In any metric space the union of any family of open set is open.

Hence, $\bigcup_{i \in I} A_i^c$ is open

$\Rightarrow \bigcap_{i \in I} A_i^c$ is open

$\Rightarrow \bigcap_{i \in I} A_i$ is closed

3) In any metric space the union of a finite number of closed set is closed.

Let $(M,d)$ be a metric space

Let $A_1, A_2, \ldots A_n$ be a closed sets in $M$.

Using De-morgan's law:

$(A_1 \cup A_2 \cup \ldots \cup A_n)^c = A_1^c \cap A_2^c \cap A_3^c \ldots \cap A_n^c$

Since, each $A_i$ is closed, then $A_i^c$ is open.

By thm:

In any Metric Space the intersection of finite number of open set is open.

$A_1^c \cap A_2^c \cap \ldots \ldots \cap A_n^c$ is open

$(A_1 \cup A_2 \cup \ldots \ldots \cup A_n)^c$ is open

$\Rightarrow A_1 \cup A_2 \cup \ldots \ldots \cup A_n$ is closed.

Closed ball:

Let $(M,d)$ be a Metric Space. Let $a \in M$. Let $r$ be any +ve real number. Then the closed ball (or) the closed Sphere with centre 'a' and radius 'r' denoted by $B_d[a,r]$ is defined by.

$$B_d[a,r] = \{x \in M \mid d(a,x) \leq r\}$$

We write $B[a,r]$ instead of $B_d[a,r]$

10M
Y.V.

1) In any metric Space every closed ball is a closed set.

Let $(M,d)$ be a metric Space

Let $B[a,r]$ be a Closed set.

To prove:

$B[a,r]^c$ is open

Case (i)

Suppose $B[a,r]^c = \phi$

$\therefore B[a,r]^c$ is open

$\therefore B[a,r]$ is closed.

Case (ii):

Suppose, $B[a,r]^c \neq \phi$

Let $x \in B[a,r]^c$

$\Rightarrow x \notin B[a,r]$

$\Rightarrow d(a,x) > r$

$\Rightarrow d(a,x) - r > 0$

Let $\boxed{r_1 = d(a,x) - r}$

To prove $B(x,r_1) \subseteq B[a,r]^c$

Let $\boxed{y \in B(x,r_1)}$

Now,

$x_m \in F_{n_0} \ \forall\ m \geq n_0$

$m, n \geq n_0 \Rightarrow x_m, x_n \in F_{n_0}$

$\Rightarrow d(x_m, x_n) < \varepsilon, \ \forall\ n, m \geq n_0$

$\therefore$ $(x_n)$ is a cauchy Sequence in M.

Since M is complete then $\exists$ a point $x \in M$

$\ni$: $(x_n) \to x$

T.P:

$x \in \bigcap_{n=1}^{\infty} F_n$

for any +ve Integer n.

$x_n, x_{n+1}, \ldots$ is a Sequence in $(F_n)$ Convergs to x

By theorem,

Let M be a metric Space.

Let $A \subseteq M$, then $x \in \bar{A} \Leftrightarrow \exists$ a Sequence $(x_n)$ in A

$\ni$: $(x_n) \to x$

$x \in \overline{F_n}$

Since, $F_n$ is closed then $F_n = \overline{F_n}$

$x \in \overline{F_n}$

$x \in \bigcap_{n=1}^{\infty} F_n$

Hence, $\bigcap_{n=1}^{\infty} F_n \neq \phi$

Conversely,

Let $(x_n)$ be a cauchy Sequence in M.

Let $F_1 = \{x_1, x_2, \ldots, x_n \ldots\}$

$F_2 = \{x_2 \ldots x_n \ldots\}$

$\ldots$

$F_n = \{x_n, x_{n+1} \ldots\}$

Proof:

Let M be a Complete metric Space.

T.P: M is not of first category.

Let $(A_n)$ be a Sequence of nowhere dense sets in M

T.P: $\bigcup_{n=1}^{\infty} A_n \neq M$

Since, M is open and $A_1$ is nowhere dense, $\exists$ an open ball Say $B_1$ of radius less than 1, $\ni$ $B_1$ is disjoint from

Let $F_1$ denote the concentric closed ball whose radius is $1/2$ times that of $B_1$ is d from $A_1$.

Now, Int $F_1$ is open and $A_2$ is nowhere dense

$\therefore$ Int $F_1$ contains an open ball $B_2$ of radius less than $1/2$ times that of $B_2$ is disjoint from $A_2$.

Let $F_2$ be the concentric closed ball whose radius is $1/2$ times of $B_2$.

Now, Int $F_2$ is open and $A_3$ is nowhere dense

$\therefore$ Int $F_2$ contains an open ball $B_3$ of radius less than $1/2$ Such that $B_3$ is disjoint from $A_3$.

Let $F_3$ be the concentric closed ball whose radius is $1/2$ times than of $B_3$

Proceeding like this we get a Sequence of non-empty closed balls $F_n$ Such that $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq$

and $d(F_n) < 1/2^n$

Hence, $(d(F_n)) \to 0$ as $n \to \infty$

**Closure :-**

Let $A$ be a Subset of Metric Space $(M,d)$. The closure of $A$, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain $A$.

Thus, $\bar{A} = \cap B \{ B$ is closed in $M$ and $A \subseteq B \}$

**Theorem:**

S.T $A$ is closed iff $A = \bar{A}$

**Proof:-**

Suppose, $A = \bar{A}$

T.p: $A$ is closed.

Since, $\bar{A}$ is closed, then $A$ is closed.

Conversely. Suppose $A$ is closed

Then the Smallest closed set containing $A$ is $A$ itself

$\therefore A = \bar{A}$.

Hence proved.

**Defn:-**

Let $(M,d)$ be a metric Space. Let $A \subseteq M$. Let $x \in M$. Then '$x$' is called a limit point (or) a cluster point (or) an accumulation point of $A$ if every open ball with centre $x$ contains atleast one point of $A$ different from '$x$'

ie) $B(x,r) \cap (A - \{x\}) \neq \phi$ for all $r > 0$

The Set of all limit points of $A$ is called the derived set of $A$ and is denoted by $D(A)$.

Since 'd' is the discrete metric space distance between any two points either 0 (or) 1.

$$\therefore d(x_n, x_m) = 0 \; \forall \; n, m \geq n_0$$

$$\therefore x_n = x_{n_0} = x \text{ (Say) } \forall \; n \geq n_0$$

$$d(x_n, x) = 0 \; \forall \; n \geq n_0$$

$$\therefore (x_n) \to x$$

Hence, M is Complete.

Theorem: 8

U.Q A Subset A of a Complete metric Space M is complete $\Leftrightarrow$ A is closed.

Proof:

Suppose, A is complete

T.p: A is closed.

ie) To prove that, A contains all its limit points

Let $x$ be a limit point of A

Then, by theorem 1, $\exists$ a Sequence $(x_n)$ in A $\ni : (x_n) \to x$

Since, A is complete, $x \in A$

$\therefore$ A contains all its limit points.

Hence A is closed.

Conversely,

Let A be a Closed Subset of M.

T.p: A is complete

Let $(x_n)$ be a Cauchy Sequence in A.

Then $(x_n)$ is a Cauchy Sequence in M and Since M is complete

$$\exists \; x \in M \ni : (x_n) \to x$$

$\therefore x \in A$.

Thus, every cauchy sequence $\{x_n\}$ in $A$ conv.. to a point in $A$.

$A$ is complete.

Hence proved !!.

3) The Metric Space $[0,1]$ and $[0,2]$ with usual metric are homeomorphic.

Define $f: [0,1] \to [0,2]$ by $f(x) = 2(x)$

Clearly, $f$ is 1-1 and onto.

Also, $f(x) = 2(x)$

$\Rightarrow f^{-1}(x) = \frac{1}{2}x$

$\therefore$ $f$ and $f^{-1}$ are both Continuous.

$\therefore$ $f$ is a homeomorphism.

4) Let $d_1$ be the usual Metric on $[0,1]$ and $d_2$ be the usual Metric on $[0,2]$.

The Map $f: [0,1] \to [0,2]$ defined by $f(x) = 2x$ is not an isometry.

Pf:

Let $x, y \in [0,1]$

Then $d_2(f(x), f(y)) = |f(x) - f(y)|$

$= |2x - 2y|$

$= 2|x-y|$

$d_2(f(x), f(y)) = 2 d_1(x,y)$

$\therefore d_1(x,y) \neq d_2(f(x), f(y))$

$\therefore f$ is not isometry.

Let, $x \in f^{-1}(B(f(x)), \epsilon)$

$\exists$, $\delta > 0$ ; $\epsilon > 0$ Such that

$B(x, \delta) \subseteq f^{-1}(B(f(x)), \epsilon))$

$\Rightarrow f(B(x,\delta) \subseteq B(f(x)), \epsilon)$

Since $x \in m_1$ is arbitrary,

$f$ is Continuous.

3) Let $m_1, m_2, m_3$ be Metric Space. If $f: m_1 \to m_2$ and $g: m_2 \to m_3$ are Continuous function. p.T $g_o f : m_1 \to m_3$ is also Continuous)

(i.e) Composition of two cts function is cts.

Let $f: m_1 \to m_2$

$g: m_2 \to m_3$ be cts function.

T.P:

$g_o f : m_1 \to m_3$ is cts

(ie) to prove that, $(g_o f)^{-1}(G)$ is open in $m_1$

Since $g$ is Continuous then $g^{-1}(G)$ is open in $m_2$.

Since $f$ is Continuous then $f^{-1}(g^{-1}(G))$ is open in $m_1$

$(g_o f)^{-1}(G)$ is open in $m_1$.

$\therefore (g_o f)$ is Continuous.

4) Let M be a Metric Space let $f: M \to R$ and $g: m \to R$ be two cts function. p.T $f+g : m \to R$ is Continuous.

Let $f: M \to R$ and $g: m \to R$ be cts function

T.P.

$(f+g)$ is Continuous

Let $(x_n) \to x$ in $m$

Since, $f$ and $g$ are cts, then

$(f(x_n)) \to f(x)$ ; $(g(x_n)) \to g(x_n)$

$(f(x_n) + g(x_n)) \to (f(x) + g(x))$

Theorem: 1

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Space. Then any constant function $f: M_1 \to M_2$ is continuous.

Proof:

Let $f: M_1 \to M_2$ be given by $f(x) = a$

Where $a \in M_2$ is a fixed element.

Let $x \in M_1$ and $\varepsilon > 0$ be given.

Then for any $\delta > 0$.

$$f(B(x,\delta)) \subseteq B(f(x), \varepsilon)$$

$$\Rightarrow f(B(x,\delta)) \subseteq B(a, \varepsilon)$$

$$\Rightarrow f(B(x,\delta)) = \{a\} \subseteq B(a, \varepsilon)$$

$\therefore f$ is continuous at $x$.

Since, $x \in M_1$ is arbitrary

$f$ is continuous.

Problems:

1) Let $f$ be a continuous real valued function defined on a Metric Space $M$. Let $A = \{x \in M \mid f(x) \geq 0\}$ prove that $A$ is closed.

$$\text{Let } A = \{x \in M \mid f(x) \geq 0\}$$

$$= \{x \in M \mid f(x) \in [0, \infty)\}$$

$$A = f^{-1}([0, \infty))$$

Also, $[0, \infty)$ is a closed subset of $R$.

Since, $f$ is continuous, $f^{-1}([0, \infty))$ is closed in $M$

$\therefore A$ is closed.

Defn:

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Spaces. Let $f: M_1 \to M_2$ be 1-1 onto Map. $f$ is Said to be an isometry if $d_1(x,y) = d_2(f(x), f(y))$ for all $x, y \in M_1$.

In otherwords, an isometry is a distance preserving Map.

$M_1$ and $M_2$ are Said to be isometric if there exists an isometry $f$ from $M_1$ onto $M_2$.

Problem:

In $R^2$ with usual Metric and $C$ with usual Metric are isometric and $f: R^2 \to C$ defined by $f(x,y) = x+iy$ is the required isometry.

Let $d_1$ denote the usual Metric on $R^2$ and $d_2$ denote the usual Metric on $C$.

Let $a = (x_1, y_1)$ and $b = (x_2, y_2) \in R^2$

Then, $d_1(a,b) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= |(x_1 - x_2) + i(y_1 - y_2)|$

$= |(x_1 + iy_1) - (x_2 + iy_2)|$

$d(a,b) = d_2(f(a), f(b))$

$\therefore f$ is an isometry

Defn:

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Space. A function $f: M_1 \to M_2$ is Said to be uniformly continuous on $M_1$ if given $\varepsilon > 0$ there exists $\delta > 0$ Such that.

$d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$

# UNIT - III

Defn: Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Spaces. Let $f : M_1 \to M_2$ be a function. Let $a \in M_1$ and $l \in M_2$. The function $f$ is said to have a limit as $x \to a$ if given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < d_1(x, a) < \delta \Rightarrow d_2(f(x), l) < \varepsilon$$

We write, $\lim_{x \to a} f(x) = l$

Defn: Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Spaces. Let $a \in M_1$. A function $f : M_1 \to M_2$ is said to be Continuous at 'a' if given $\varepsilon > 0$ $\exists\ \delta > 0$ such that,

$$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$$

$f$ is said to be Continuous if it is Continuous at every point of $M_1$.

Note:

(1) $f$ is Continuous at 'a' $\Leftrightarrow \lim_{x \to a} f(x) = f(a)$

(2) The Condition $d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon$ can be rewritten as (i) $x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$

(ii) $f(B(a, \delta)) \subseteq B(f(a), \varepsilon)$

$(f+g)x_n \to (f+g)x$

$f+g$ is Continuous.

2.2.5m

Thm 4.3

Let $(m_1, d_1)$ and $(m_2, d_2)$ be two Metric space. A function $f: m_1 \to m_2$ is continuous iff $f^{-1}(f)$ is closed in $m_1$. Whenever $f$ is closed in $m_2$.

Proof:

Suppose $f: m_1 \to m_2$ is Continuous

Let $f \subseteq m_2$ (closed in $m_2$)

$\therefore f^c$ is open in $m_2$.

By the thm".

Let $(m_1, d_1)$ and $(m_2, d_2)$ be two Metric space. Let $f: m_1 \to m_2$ is cts. iff $f^{-1}(G)$ is open in $m_1$. Whenever $G$ is open in $m_2$".

$f^{-1}(f^c)$ open in $m_1$.

$\therefore [f^{-1}(f^c)] = [f^{-1}(f)]^c$ is open in $m_1$.

$[f^{-1}(f)]^c$ is open in $m_1$.

Conversely.

Suppose $f^{-1}(f)$ is closed in $m_1$ Whenever $f$ is closed in $m_2$.

T.p: $f$ is Continuous.

(ie) to prove that $f^{-1}(G)$ is open in $m_1$.

Let $G$ be an open in $m_2$.

$G^c$ is closed in $m_2$

$\therefore f^{-1}(G^c)$ is closed in $m_1$

$[f^{-1}(G)]^c$ is closed in $m_1$ $[\because f^{-1}(G^c) = [f^{-1}(G)]^c]$

$f^{-1}(G)$ is open in $m_1$

$\therefore f$ is Continuous.

1) Prove that $f$ is Continuous iff inverse image of every open set is open.

Statement:

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Space.

Let $f: M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in $M_1$ Whenever $G$ is open in $M_2$.

Proof:

Suppose $f$ is Continuous

Let $G$ be an open Set in $M_2$.

To prove, $f^{-1}(G)$ is open in $M_1$

If $f^{-1}(G) = \phi$, then it is open

If $f^{-1}(G) \neq \phi$, then $x \in f^{-1}(G)$

$f(x) \in G$

Since $G$ is open, then there exists an open ball $B(f(x), \varepsilon)$

$\ni$: $B(f(x), \varepsilon) \subseteq G$

Then there exists an open ball

$B(x, \delta)$ Such that $(B(x, \delta)) \subseteq B(f(x), \varepsilon)$

$\Rightarrow f(B(x, \delta)) \subseteq G$

$\Rightarrow B(x, \delta) \subseteq f^{-1}(G)$

$\therefore x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open.

Conversely:

Suppose $G$ is $f^{-1}(G)$ is open in $M_1$ Whenever $G$ is open in $M_2$.

To prove: $f$ is Continuous

Let $x \in M$

$\Rightarrow B(f(x), \varepsilon)$ is open Set in $M_2$.

$\Rightarrow f^{-1}(B(f(x)), \varepsilon)$ is open in $M_1$

Thm 4.1

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is continuous at $a$ iff $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$

Suppose $f$ is continuous at 'a'.

To prove: $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$

Let $(x_n)$ be a Sequence in $M_1$, Such that $(x_n) \to a$

(i.e) To prove,

$$(f(x_n)) \to f(a)$$

Let $\epsilon > 0$ be given, $\exists$ $\delta > 0$, Such that

$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \epsilon$

Since, $(x_n) \to a$, the $\exists$ $\delta > 0$ and there is a +ve integer $n \geq n_0$

$$\Rightarrow d(x_n, a) < \delta \Rightarrow d_2(f(x_n), f(a)) < \epsilon$$

$$\Rightarrow (f(x_n)) \to f(a)$$

Conversely,

let $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$

To prove,

$f$ is Continuous at $a$ suppose $f$ is not Continuous at 'a'.

Then there exists $\epsilon > 0$ and $\delta > 0$

$$f(B(a, \delta)) \not\subset B(f(a), \epsilon)$$

$$\Rightarrow f(B(a, 1/n)) \not\subset B(f(a), \epsilon)$$

Choose, $x_n \in B(a, 1/n)$

$$\Rightarrow f(x_n) \notin B(f(a), \epsilon)$$

$d(x_n, a) < 1/n$

$$d_2(f(x_n), f(a)) \geq \epsilon$$

$\therefore (x_n) \to a$, $(f(x_n)) \not\to f(a)$

Which is Contradiction to our assumption.

$\therefore f$ is Continuous at $a$.

$x = y - xy$

$x + xy = y$

$x(1+y) = y$

$x = y/1+y \in [0,1)$

$\therefore f$ is onto

Hence $f$ is homeomorphic

pbm:1

P.T $f : [0,1] \to R$ defined by $f(x) = x^2$ is uniformly continuous on $[0,1]$

Let $\varepsilon > 0$ be given

Let $x, y \in [0,1]$

clearly, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$\Rightarrow |x^2 - y^2|$

$\Rightarrow |x+y||x-y|$

Since $x \leq 1$ and $y \leq 1$

Then, $|x-y| < \delta \Rightarrow 2|x-y|$

$|x-y| < 1/2\,\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon \quad (\because \delta = 1/2\,\varepsilon)$

$\therefore f$ is uniformly continuous on $[0,1]$

pbm:2

Prove that the function $f : (0,1) \to R$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Let $\varepsilon > 0$ be given. Suppose $\exists$ $\delta > 0$ $\ni$: $|x-y| < \delta \Rightarrow$ $|f(x) - f(y)| < \varepsilon$

Take $x = y + \frac{1}{2}\delta$

Clearly $|x-y| = 1/2\,\delta < \delta$

$\therefore |f(x) - f(y)| < \varepsilon$

$\left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$

Note:

* If $f: M_1 \to M_2$ is uniformly continuous in $M_1$ then $f$ is continuous at every point of $M_1$.

P.T $\mathbb{C}$ with usual Metric is Complete

Proof:

Let $(z_n)$ be a Cauchy Sequence in $\mathbb{C}$.

Let $z_n = x_n + iy_n$, Where $x_n, y_n \in \mathbb{R}$

We prove that, $(x_n)$ and $(y_n)$ are Cauchy Sequences.

Let $\varepsilon > 0$ be given.

Since $(z_n)$ is a Cauchy Sequence, then $\exists$ a +ve Integer

$\ni$: $|z_n - z_m| < \varepsilon \ \forall \ n, m \geq n_0$

$\therefore |x_n - x_m| \leq |z_n - z_m|$ and

$|y_n - y_m| \leq |z_n - z_m|$

Hence, $|x_n - x_m| < \varepsilon \ \forall \ n, m \geq n_0$

$|y_n - y_m| < \varepsilon \ \forall \ n, m \geq n_0$

$\therefore (x_n)$ and $(y_n)$ are Cauchy Sequence in $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $\exists\ x, y \in \mathbb{R}$ $\ni$: $(x_n) \to x$ and $(y_n) \to y$

Let, $z = x + iy$

T.P: $(z_n) \to z$

We have, $|z_n - z| = |(x_n + iy_n) - (x + iy)|$

$= |(x_n - x) + i(y_n - y)|$

$|z_n - z| \leq |(x_n - x)| + |y_n - y|$ — (1)

Let $\varepsilon > 0$ be given

Since $(x_n) \to x$ and $(y_n) \to y$ then $\exists$ a +ve integer $n_1$ and $n_2$.

Such that,

$|x_n - x| < \frac{1}{2}\varepsilon \ \forall n \geq n_1$

and $|y_n - y| < \frac{1}{2}\varepsilon \ \forall n \geq n_2$

let $n_3 = \max\{n_1, n_2\}$

(1) $\Rightarrow |x_n - x| \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \ \forall n \geq n_3$

$|x_n - x| < \varepsilon \ \forall n \geq n_3$

$\therefore (x_n) \to x$

$\therefore$ C is Complete.

) A subset A of a Complete Metric Space M is Complete iff A is closed.

Suppose, A is Complete.

T.P: A is closed.

ie) To prove that, A contains all its limit points.

Let '$x$' be a limit point of A

By theorem,

There exists a Sequence $(x_n)$ in A.

Such that $(x_n) \to x$.

Since, A is Complete, $x \in A$

$\therefore$ A Contains all its limit points.

Hence A is closed.

Conversely, let A be a closed subset of M.

Let $(x_n)$ be a Cauchy sequence in A.

Then $(x_n)$ be a Cauchy sequence in M and Since M is Complete, $\exists$ $x \in M$ Such that $(x_n) \to x$.

Thus, $(x_n)$ is a Sequence in A Converging to $x$.

$\therefore x \in \overline{A}$

Since, A is closed, then $A = \overline{A}$

Thm 4.9

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Space. Then $f: m_1 \to m_2$ is Continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq m_1$

Suppose $f$ is Continuous.

T.P: $f(\bar{A}) \subseteq \overline{f(A)}$

Let $A \subseteq M_1$

$f(A) \subseteq m_2$

$\because f$ is cts, $f^{-1}(\overline{f(A)})$ is closed in $m_1$

$A \subseteq f^{-1}(\overline{f(A)})$

$\bar{A} \subseteq f^{-1}(\overline{f(A)})$ $(\because A \subseteq \bar{A})$

$f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$

Conversely.

$f(\bar{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$

T.P: $f$ is Continuous

(ie) $f^{-1}(f)$ is closed in $m_1$.

By hypothesis, $F(\overline{f^{-1}(f)}) \subseteq \overline{f(f^{-1}(f))}$

$\subseteq \bar{f}$

$f(\overline{f^{-1}(f)}) \subseteq f$ $(\because f \subseteq \bar{f})$

$\overline{f^{-1}(f)} \subseteq f^{-1}(f)$

$F^{-1}(F) \subseteq \overline{F^{-1}(F)}$

$\because f^{-1}(f)$ is closed in $M_1$. Hence $F$ is Continuous.

6) If $f: R \to R$ and $g: R \to R$ are both cts function on $R$ and if $h: R^2 \to R^2$ is defined by $h(x, y) = (f(x), g(y))$ P.T $h$ is Continuous on $R^2$.

Let $f: R \to R$ and $g: R \to R$ be Continuous function on $R$.

T.p:

$h: R^2 \to R^2$ is cts

(i.e) T.p $(h(x_n, y_n)) \to h(x, y)$

5) P.T the function $f: R \to R$ defined by $f(x) = \sin x$ is uniformly Continuous on $R$.

Let $x, y \in R$ and $x > y$

$\sin x - \sin y = (x-y)\cos z$, Where $x > z > y$

$\therefore |\sin x - \sin y| = |(x-y)\cos z|$

$= |x-y|\,|\cos z|$

$|\sin x - \sin y| \leq |x-y| \quad (\because |\cos z| \leq 1)$

For given $\varepsilon > 0$, We choose $\delta = \varepsilon$,

We've $|x-y| < \delta \Rightarrow |f(x) - f(y)|$

$\Rightarrow |\sin x - \sin y| < \varepsilon$

$\therefore f(x) = \sin x$ is uniformly Continuous on $R$.

Let $((x_n, y_n)) \to (x, y)$ in $R^2$

T.P: $(h(x_n, y_n)) \to h(x,y)$

Since $f$ and $g$ are cts

$(f(x_n)) \to f(x)$ and $(g(y_n)) \to g(y)$

$(f(x_n), g(y_n)) \to (f(x), g(y))$

$(h(x_n, y_n)) \to h(x,y)$

$\therefore$ h is Continuous on $R^2$

Ex: 4 The Metric space $(0,1)$ and $(0,\infty)$ with usual Metric are homeomorphic.

Define $f: (0,1) \to (0,\infty)$

by $f(x) = \frac{x}{1-x}$

Clearly, ~~the usual Metric are ct~~ (i.e) f is cts and $f^{-1}$ is also Continuous.

No prove that f is bijective.

(i) f is 1-1

$$f(x) = f(y)$$

$$\frac{x}{1-x} = \frac{y}{1-y}$$

$$x(1-y) = y(1-x)$$

$$x - xy = y - xy$$

$$\boxed{x = y}$$

$\therefore$ f is 1-1

(ii) f is onto

$$f(x) = y$$

$$\frac{x}{1-x} = y$$

$$x = y(1-x)$$

$$\left| \frac{1}{y+1/2\,\delta} - \frac{1}{y} \right| < \epsilon$$

$$\left| \frac{\delta}{2(y+1/2\,\delta)y} \right| < \epsilon$$

$$\therefore \frac{\delta}{(2y+\delta)y} < \epsilon$$

This inequality cannot be true for all $y \in (0,1)$

**Defn:**

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two Metric Space. A function $f: M_1 \to M_2$ is called a homeomorphism if.

(i) $f$ is 1-1 and onto

(ii) $f$ is continous

(iii) $f^{-1}$ is Continuous

$M_1$ and $M_2$ are said to be homeomorphic if there exist a homeomorphism $f: M_1 \to M_2$

**Defn:**

A function $f: M_1 \to M_2$ is Said to be an open Map if $f(G)$ is open in $M_2$ for every open set $G$ in $M_1$.

ie) $f$ is an open Map if the image of an open set in $M_1$. is an open Set in $M_2$.

$f$ is called a closed Map if $f(F)$ is closed in $M_2$ for every closed set $F$ in $M_1$.

**Note:**

* Let $f: M_1 \to M_2$ be a 1-1 and onto function. Then $f^{-1}$ is Continuous iff $f$ is an open Map.

Similarly, $f^{-1}$ is Continuous iff $f$ is a Closed Map

Defn:

Let $(M,d)$ be a Metric Space. $M$ is said to be Connected if $M$ cannot be represented as the union of two disjoint non-empty open sets.

If $M$ is not Connected it is said to be disconnected.

Example:

Let $M = [1,2] \cup [3,4]$ with usual Metric. Then $M$ is disconnected.

Let $M = [1,2] \cup [3,4]$

Let $A_1 = [1,2]$

Then $A_1 = [1,2]$

$A_1 = (1/2, 5/2) \cap M$

$[1,2]$ is open in $M$

Similarly, $[3,4]$ is open in $M$

$\therefore$ $[1,2]$ and $[3,4]$ are open in $M$

Thus, $M$ is the union of two disjoint non-empty open sets namely $[1,2]$ and $[3,4]$

Hence $M$ is disconnected.

1) P.T if $f$ is a non-constant real valued continuous function on $R$, then the range of $f$ is uncountable.

W.K.T $R$ is Connected.

Since $f$ is a Continuous function on $R$,

$f(R)$ is Connected subset of $R$.

$\therefore$ $f(R)$ is an interval in $R$.

Also, Since $f$ is a non-constant function the interval.

$f(R)$ contains More than one point

$\therefore f(R)$ is uncountable

(ie) The range of $f$ is uncountable.

2) Any continuous image of a Connected set is Connected.

Statement:

Let $M_1$ be a Connected Metric Space. Let $M_2$ be any Metric Space. Let $f: M_1 \to M_2$ be Continuous function. Then $f(M_1)$ is a Connected subset of $M_2$.

Proof: Let $f(M_1) = A$ So that $f$ is a function from $M_1$ onto $A$.

T.P: A is Connected

Suppose, A is not Connected.

Then, there exists a proper non-empty Subset B of A Which is both open and closed in A.

$\therefore f^{-1}(B)$ is a proper non-empty subset of $M_1$ Which is both open and closed in $M_1$.

Hence $M_1$ is not connected. Which is a $\Rightarrow\Leftarrow$

Hence, A is Connected.

3) A subspace of R is Connected iff it is an interval.

Pf: Let A be a Connected Subset of R.

Suppose, A is not an interval.

Then $\exists$ $a, b, c \in R$ $\ni$: $a < b < c$ and $a, c \in A$ but $b \notin A$

Now, Since $A$ is an interval

We have: $[x,z] \subseteq A$

ie) $[x,z] \subseteq A_1 \cup A_2$

Every element of $[x,z]$ is either in $A_1$ or in $A_2$.

Now, let $y = \text{l.u.b} \{[x,z] \cap A_1\}$

Clearly $x \leq y \leq z$

Hence $y \in A$

Let $\varepsilon > 0$ be given. Then $\exists\, t \in [x,z] \cap A_1$

Such that

$$y - \varepsilon < t \leq y$$

$$\therefore (y-\varepsilon, y+\varepsilon) \cap ([x,z] \cap A_1) \neq \phi$$

$$\therefore y \in \overline{[x,z] \cap A_1}$$

$$y \in [x,z] \cap A_1$$

$$\therefore \boxed{y \in A_1} \quad \text{— (1)}$$

Again, $y+\varepsilon \in A_2$ $\forall\, \varepsilon > 0$ Such that $y + \varepsilon \leq z$

$$\therefore y \in \overline{A_2}$$

$$\boxed{y \in A_2} \quad (\because A_2 \text{ is closed}) \quad \text{— (2)}$$

$$\therefore y \in A_1 \cap A_2 \quad (\because \text{by (1) and (2)})$$

Which is a $\Rightarrow \Leftarrow$

Since $A_1 \cap A_2 = \phi$

Hence $A$ is Connected.

State and prove Intermediate Value theorem

Statement:

Let $f$ be a real valued continuous function defined on an interval $I$. Then $f$ takes every value between any two value it assume.

Proof:

Let $a, b \in I$ ; Let $f(a) \neq f(b)$

Without loss of generality, We may assume that $f(a) < f(b)$

Let $c$ be such that $f(a) < c < f(b)$

The interval $I$ is the connected subset of $R$.

($\because$ A subset of $R$ is connected iff it is an interval)

Since By thm,

" Let $M_1$ be a connected Metric space. Let $M_2$ be any Metric space. Then $f: M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of $M_2$".

$\Rightarrow f(x)$ is also a connected subset of $A$.

$\Rightarrow f(x)$ is an interval

$\Rightarrow f(a), f(b) \in f(x)$

$[f(a), f(b)] \subseteq f(I)$

$c \in f(I)$

$\Rightarrow \boxed{c = f(x)}$ for some $x \in I$

Since $f$ is onto, $A$ and $B$ are non-empty.

Clearly, $A \cup B = M$; $A \cap B = \phi$

Then $M = A \cup B$, where $A$ and $B$ are disjoint open sets.

$M$ is disconnected which is a $\Rightarrow \Leftarrow$.

(contradiction)

Hence there does not exists a continuous function.

Conversely,

Suppose $M$ is disconnected. Then there exists a disjoint open sets $A$ and $B$ in $M$.

Such that $M = A \cup B$

Define $f : m \to \{0,1\}$ by,

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

$\Rightarrow$ clearly $f$ is onto

$$f^{-1}(\phi) = \phi$$

$$f^{-1}(\{0\}) = A,\ f^{-1}(\{1\}) = B\ ;\ F^{-1}(\{0,1\}) = M$$

Thus the inverse image of open sets, $\{0,1\}$ is open in

Hence $f$ is continuous and there exists a continuous function.

$f : m \to \{0,1\}$ $\Rightarrow \Leftarrow$ $M$ is connected.

UNIT - $\overline{IV}$

Thm :-

M is connected iff every continuous function $f: M \to \{0,1\}$ is not onto.

Statement :

A Metric Space M is Connected iff there does not exists a Continuous function $f$ from M onto the discrete Metric Space $\{0,1\}$

Proof :

Suppose there exists a cts onto function

$f: m \to \{0,1\}$

Since $\{0,1\}$ is discrete, $\{0\}$ and $\{1\}$ are open.

$\therefore A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are open in M

Let $A_1 = (-\infty, b) \cap A$

$A_2 = (b, \infty) \cap A$

Since, $(-\infty, b)$ and $(b, \infty)$ are open in $\mathbb{R}$,

$A_1$ and $A_2$ are open sets in $A$.

Also, $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = A$.

Further, $a \in A_1$ and $c \in A_2$

Hence, $A_1 \neq \phi$ and $A_2 \neq \phi$

Thus, $A$ is the union of two disjoint non-empty open sets $A_1$ and $A_2$.

Hence $A$ is not connected. Which is a $\Rightarrow \Leftarrow$

Hence $A$ is an interval.

Conversely.

Let $A$ be an interval

T.P: $A$ is connected.

Suppose, $A$ is not connected.

Let $A = A_1 \cup A_2$, Where $A_1 \neq \phi$, $A_2 \neq \phi$, $A_1 \cap A_2 = \phi$ and $A_1$ and $A_2$ are closed sets in $A$.

choose $x \in A_1$ and $z \in A_2$.

Since, $A_1 \cap A_2 = \phi$

We have $x \neq z$

Without loss of generality,

We assume that $x < z$.

# Unit - V

**Defn:**

Let $M$ be a metric space. A family of open sets $\{G_\alpha\}$ in $M$ is called an open cover for $M$ if $\bigcup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a subcover.

A metric space $M$ is said to be compact if every open cover for $M$ has a finite subcover.

ie) For each family of open sets $\{G_\alpha\}$ such that $\bigcup G_\alpha = M$, there exists a finite sub family $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = M$.

**Defn:**

A family $\mathcal{J}$ of subsets of a set $M$ is said to have the finite intersection property if any finite members of $\mathcal{J}$ have non-empty intersection.

**Defn:**

A metric space $M$ is said to be totally bounded if for every $\varepsilon > 0$ $\exists$ a finite number of elements $x_1, x_2, \dots, x_n \in M$ such that

$$\boxed{B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \dots \cup B(x_n, \varepsilon) = M}$$

A non-empty subset $A$ of a metric space $M$ is said to be totally bounded if the subspace $A$ is a totally bounded metric space.

Defn:

Let $(x_n)$ be sequence in a metric space M. Let $n_1 < n_2 < \dots < n_k < \dots$ be an increasing sequence of +ve integers. Then $(x_{n_k})$ is called a subsequence of $(x_n)$

---

Defn:

A metric space M is said to be sequentially compact if every sequence in M has a convergent subsequence.

---

Thm:1

1) P.T R with usual metric is not compact.

proof:

Consider the family of open intervals $\{(-n, n) / n \in N\}$.

This is a family of open sets in R.

Clearly $\bigcup_{n=1}^{\infty} (-n, n) = R$.

$\therefore \{(-n, n) / n \in N$ is an open cover for R.

and this open cover has no finite subcover.

$\therefore$ R is not compact.

---

2) P.T $(0, 1)$ with usual metric is not compact.

proof:

Consider the family of open intervals $\{(1/n, 1) / n = 2, 3 \dots\}$

Clearly $\boxed{\bigcup_{n=2}^{\infty} (1/n, 1) = (0, 1)}$

say, $f^{-1}(G_{\alpha_1}), f^{-1}(G_{\alpha_2}) \cdots f^{-1}(G_{\alpha_n})$

$$f^{-1}(G_{\alpha_1}) \cup f^{-1}(G_{\alpha_2}) \cup \cdots \cup f^{-1}(G_{\alpha_n}) = M_1$$

$$f^{-1}\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) = M_1$$

$$\bigcup_{i=1}^{n} G_{\alpha_i} = f M_1 = M_2.$$

$$\bigcup_{i=1}^{n} G_{\alpha_i} = M_2.$$

$\therefore$ $G_{\alpha_1}, G_{\alpha_2}, \cdots G_{\alpha_n}$ is an open cover for $M_2$.

This gives a finite subcover for $M_2$.

$\therefore$ $M_2$ is compact.

---

Any compact subset $A$ of a metric space $M$ is bounded.

<u>Proof:</u>

Let $x_0 \in A$.

Consider $\{B(x_0, n) \mid n \in N\}$

Clearly $\bigcup_{n=1}^{\infty} B(x_0, n) = M$.

$$\therefore \bigcup_{n=1}^{\infty} B(x_0, n) \supseteq A.$$

Since $A$ is compact there exists a finite sub family say,

$B(x_0, n_1), B(x_0, n_2) \cdots B(x_0, n_k)$ such that,

$$\bigcup_{i=1}^{k} B(x_0, n_i) \supseteq A.$$

Let $n_0 = \max\{n_1, n_2, \cdots n_k\}$

Then $\bigcup_{i=1}^{k} B(x_0, n_i) = B(x_0, n_0)$

$\therefore$ $B(x_0, n_0) \supseteq A$.

These finite number of $G_\alpha$ is together with $G_{\alpha_1}$ covers $[a, c]$

$\therefore c \in S$.

To prove:

$c = b$.

Suppose, $c \neq b$.

Then choose $x_2 \in [a, b]$ such that $x_2 > c$

$[c, x_2] \subseteq G_{\alpha_1}$

As before $[a, x_2]$ can be covered by a finite number of $G_\alpha$ is

Hence $x_2 \in S$.

But $x_2 > c$ which is a $\delta$

Since $c$ is l.u.b of $S$

$\therefore c = b$

$\therefore [a, b]$ can be covered by a finite number of $G_\alpha$ is.

$\therefore [a, b]$ is a compact subset of $R$.

$\therefore \left(\bigcup_{\alpha \in I} G_\alpha\right)^c = M^c$

$\therefore \bigcap_{\alpha \in I} G_\alpha^c = \phi$

Since, $G_\alpha$ is open, $G_\alpha^c$ is closed for each $\alpha$

$\therefore \mathcal{F} = \{G_\alpha^c / \alpha \in I\}$ is a family of closed sets whose intersection is empty.

Hence, by hypothesis, this family of closed sets doesn't have the finite intersection property.

Hence, there exists a finite sub-collection of $\mathcal{F}$ say,

$\{G_1^c, G_2^c, \ldots, G_n^c\}$ such that $\bigcap_{i=1}^{n} G_i^c = \phi$.

$\therefore \left(\bigcup_{i=1}^{k} G_i\right)^c = \phi$.

$\bigcup_{i=1}^{n} G_i = M$.

$\therefore \{G_1, G_2, \ldots, G_n\}$ is a finite subcover of the given open cover.

Hence $M$ is compact.

**Theorem:**

Any compact metric space is totally bounded.

**Proof:**

Let $M$ be a compact metric space.

Then $\{B(x, \epsilon) / x \in M\}$ is an open cover for $M$.

Since, $M$ is compact this open cover has a finite subcover say,

$B(x_1, \epsilon), B(x_2, \epsilon), \ldots, B(x_n, \epsilon)$

We know that $B(x_0, r_0)$ is a bounded set and a subset of a bounded set is bounded

Hence $A$ is bounded.

## Heine Borel theorem.

**Statement:**

Any closed interval $[a,b]$ is a compact. of $R$.

**Proof:**

Let $\{G_\alpha / \alpha \in I\}$ be a family of open set in $R$ such that,

$$\bigcup_{\alpha \in I} G_\alpha \supseteq [a,b]$$

Let $S = \{x / x \in [a,b]\}$ and $[a,x]$ can be covered by a finite number of $G_\alpha$ is clearly $a \in S$, and hence $S \neq \phi$. Also $S$ is bounded above by $b$.

Let $c$ denote the l.u.b of $S$ clearly $c \in [a,b]$.

$\therefore c \in G_{\alpha_1}$ for some $\alpha_1 \in I$ since $G_{\alpha_1}$ is open there exists $\varepsilon > 0$ such that $(c-\varepsilon, \varepsilon+c) \subseteq G_{\alpha_1}$

Choose, $x_1 < c$ ; $[x_1, c] \subseteq G_{\alpha_1}$

Now since $x_1 < c$ ; $[a, x_1]$ can be covered. by finite number of $G_\alpha$ is

Hence $(x_n)$ converges to some point $x \in [a,b]$

Thus, every cauchy sequence $(x_n)$ in $R$. converges to some point $x$ in $R$.

Hence $R$ is complete.

Continuous image of a compact metric space is compact.

Statement:

Let $f$ be a continuous mapping from a compact metric space $m_1$ to any metric space $m_2$. Then $f(m_1)$ is compact.

Proof:

Without loss of generality we may assume that

$$\boxed{f(m_1) = m_2}$$

Let $\{G_\alpha\}$ be a family of open set in $m_2$ such that $\cup G_\alpha = m_2$

$$\therefore \cup G_\alpha = f(m_1)$$

$$f^{-1}(\cup G_\alpha) = m_1$$

$$\cup(f^{-1}(G_\alpha)) = m_1$$

Also, since $f$ is continuous.

$f^{-1}(G_\alpha)$ is open in $m_1$ for each $\alpha$.

$\{f^{-1}(G_\alpha)\}$ is an open cover for $m_1$

Since $m_1$ is compact then the open cover has a finite subcover.

$\therefore \{(1/n, 1) \mid n = 2, 3 \cdots\}$ is an open cover for $(0,1)$ and this open cover has no finite subcover.

Hence $(0,1)$ is not compact.

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Thm: A metric space $M$ is compact iff any family of closed sets with finite intersection property has non-empty intersection.

Proof:

T.P: $\cap A_\alpha \neq \phi$

Suppose $\cap A_\alpha = \phi$ then $(\cap A_\alpha)^c = \phi^c$

$\therefore \cup A_\alpha^c = M.$

Also since each $A_\alpha$ is closed, $A_\alpha^c$ is open

$\therefore \{A_\alpha^c\}$ is an open cover for $M$. Since, $M$ is compact this open cover has a finite subcover say $A_1^c, A_2^c, \ldots A_n^c$

$\therefore \bigcup_{i=1}^{n} A_i^c = M.$

$\therefore (\bigcap_{i=1}^{n} A_i)^c = M$

$\therefore \bigcap_{i=1}^{n} A_i = \phi$ which is a $\Rightarrow\Leftarrow$ to the finite intersection ~~proofect~~ property

$\boxed{\therefore \cap A_\alpha \neq \phi}$

Conversely, suppose that each family of closed sets in $M$ which finite intersection property has non-empty intersection.

T.P: $M$ is compact, let $\{G_\alpha \mid \alpha \in I\}$ be an open cover for $M$.

$\therefore \bigcup_{\alpha \in I} G_\alpha = M.$

$\therefore M = B(x_1,\varepsilon) \cup B(x_2,\varepsilon) \cup \dots \cup B(x_n,\varepsilon)$

$\therefore$ M is totally bounded.

Thm:

Let $(x_n)$ be a cauchy sequence in a metric space M. If $(x_n)$ has a subsequence $(x_{n_k})$ converging to $x$, then $(x_n)$ converges to $x$.

Proof:

Let $\varepsilon > 0$ be given.

Since $(x_n)$ is a cauchy sequence, $\exists$ a +ve integer $m_1$ such that.

$d(x_n, x_m) < \frac{1}{2}\varepsilon \;\forall\; n, m \geq m_1 \rightarrow (1)$

Also since $(x_{n_k}) \rightarrow x$, $\exists$ a +ve integer $m_2$ such that $d(x_{n_k}, x) < \frac{1}{2}\varepsilon \;\forall\; n_k \geq m_2 \rightarrow (2)$

let $m_0 = \max\{m_1, m_2\}$

and fix $n_k \geq m_0$

Then, $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$

$< \varepsilon/2 + \varepsilon/2 \;\forall\; n \geq m_0$

$d(x_n, x) < \varepsilon \;\forall\; n \geq m_0$

Hence $(x_n) \rightarrow x$.

Thm: P.T R with usual metric is complete.

Proof:

let $(x_n)$ be a cauchy sequence in R.

Then $(x_n)$ is a bounded sequence and hence is contained in a closed interval $[a,b]$.

Now $[a,b]$ is compact and

Hence is complete